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Quantum Transport Equations for a Scalar Field

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Abstract

We derive quantum Boltzmann equations from Schwinger-Dyson equations in gradient expansion for a weakly coupled scalar field theory with a spatially varying mass. We find that at higher order in gradients a full description of the system requires specifying not only an on shell distribution function but also a finite number of its derivatives, or equivalently its higher moments. These derivatives describe quantum coherence arising as a consequence of localization in position space. We then show that in the limit of frequent scatterings coherent quantum effects are suppressed, and the transport equations reduce to the single Boltzmann equation for particle density, in which particles flow along modified semiclassical trajectories in phase space.

1 Introduction

In this letter we present a controlled derivation of dynamical transport equations for a simple complex scalar theory

$$\mathcal{L} = (\partial_\mu \phi)^\dagger (\partial^\mu \phi) - m^2(\vec{x}, t)\phi^\dagger \phi + \mathcal{L}_{\rm int}\,, \tag{1}$$

where the mass represents coupling to a classical background field which varies in space and time, and $\mathcal{L}_{\rm int}$ denotes interactions. For definiteness we consider here the simple quartic interaction $\mathcal{L}_{\rm int} = -\lambda(\phi^*\phi)^2/4$. In particular we study the effect of the breakdown of translational invariance with a treatment of the varying background mass at nontrivial order in gradients.

The motivation for this work comes from electroweak baryogenesis at a first order phase transition [1], where one needs to model the departure from thermal equilibrium induced at the phase boundary of a growing bubble of the broken phase. No systematic treatment of such plasma dynamics, adequate to incorporate the crucial CP violating effects, is yet available. This paper is the second in a series in which we attempt to provide a systematic formalism for the treatment of this problem. Including higher order gradients is required since, in realistic cases, the relevant CP violating effects occur typically at higher order in gradients. The formalism we are developing is general and may be applied to other problems which involve analogous physics.

Our treatment of the problem is based on the out-of-equilibrium closed time contour (CTC) formalism. In our derivation we assume a weak coupling limit and the semiclassical approximation. The semiclassical condition, $kL \gg 1$, states that the de Broglie wave length in the relevant direction must be large in comparison to the corresponding scale of variation of the external field L. Since electroweak bubble walls are believed to be thick: $L \sim (10 - 20)T^{-1}$, and relevant particle species are weakly coupled, these constraints are satisfied for most of particles in the electroweak plasma. Our treatment does not apply to reflecting particles, for which $kL \sim 1$.

In a recent paper [2] we have analyzed the propagator of the scalar field theory in Eq. (1) to non-trivial order in gradient expansion. The main implication of this analysis is that the quasiparticle picture, in which the plasma is treated as a collection of one-particle excita-

tions with a given dispersion relation, breaks down. The reason is that in the absence of translational invariance, states localized in coordinate space mix coherently in momentum space. As a consequence, a self-consistent description of transport of plasma excitations requires additional equations which encode information about quantum coherence. In this letter we derive such a set of quantum transport equations. Further we show that in the limit of frequent scatterings when $\Gamma_{qc} L \gg 1$, where Γ_{qc} is the relevant scattering rate, the terms describing coherent quantum effects are suppressed and can be neglected. This can be understood simply as the result of the scattering projecting onto the local semiclassical (quasiparticle) states. In this limit we recover a Boltzmann equation which describes collisions and canonical flow on non-trivially modified semiclassical trajectories in phase space. In this letter we restrict our analysis to a simple scalar theory, but we expect that the structure of the transport equations derived is generic. In the discussion section we briefly outline the relevance of our findings for computations of baryon production at the electroweak scale.

2 Kadanoff-Baym Equations

The basic quantity in our derivation is the 2-point Green function in the out-of-equilibrium field theory. It is most conveniently defined in the Schwinger-Keldysh formalism [3] on a closed time contour (CTC)

$$G_{\mathcal{C}}(x,y) = -i \left\langle T_{\mathcal{C}} \left[\phi(x)\phi^\dagger(y) \right] \right\rangle , \tag{2}$$

where $T_{\mathcal{C}}$ defines time ordering along the contour $\mathcal{C}$ which starts at some t_0, often taken to be at $-\infty$, goes to $+\infty$, and then back to t_0. The two point function $G_{\mathcal{C}}(x,y)$ obeys the contour Schwinger-Dyson equation (see Fig. 1):

$$G_{\mathcal{C}}(x,y) = G^0_{\mathcal{C}}(x,y) + \int_{\mathcal{C}} dx' \int_{\mathcal{C}} dx''\, G^0_{\mathcal{C}}(x,x')\Sigma_{\mathcal{C}}(x',x'')G_{\mathcal{C}}(x'',y) \, , \tag{3}$$

where $\Sigma_{\mathcal{C}}$ is the self-energy and $G^0_{\mathcal{C}}$ is the free particle (tree level) propagator. In order to solve $G_{\mathcal{C}}(x,y)$ from (3) some external information about the self energy function $\Sigma_{\mathcal{C}}$ must be provided. In general this means coupling infinitely many new equations to (3), leading to the quantum generalization of the BBGKY hierarchy. In the weak coupling limit it is natural to

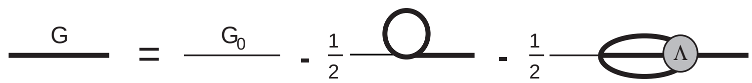

Figure 1: Diagrammatic representation of the Schwinger-Dyson equation for the scalar theory with $\mathcal{L}_{\rm int} = -\lambda(\phi^*\phi)^2/4$.

truncate the hierarchy by substituting all higher than 2-point functions by the perturbative value for the corresponding interaction vertex. For the scalar theory in Eq. (1) the simplest such truncation consists of approximating the four point function by the simplest four point vertex linear in the quartic coupling λ. As a consequence the two-point self-energies are evaluated at two loops (*cf.* Fig 1) and the resulting dynamical equations are then truncated at the order λ^2 assuming the weak coupling limit.

The complex time ordering formulation in (2-3) can be conveniently expressed in terms of the usual time ordered Green functions along the real axis. If we define

$$
\begin{aligned}
G^{>}(x,y) &= -i\langle\phi(x)\phi(y)^\dagger\rangle && (4)\\
G^{<}(x,y) &= -i\langle\phi(y)^\dagger\phi(x)\rangle && (5)\\
G^{r}(x,y) &= \theta(x_0-y_o)(G^{>}(x,y)-G^{<}(x,y)) && (6)\\
G^{a}(x,y) &= -\theta(y_0-x_0)(G^{>}(x,y)-G^{<}(x,y)), && (7)
\end{aligned}
$$

where $G^{r,a}$ are the retarded and advanced functions, we can write (3) as an equivalent set of *Kadanoff-Baym* equations [4]:

$$
\begin{aligned}
(G_0^{-1}-\Sigma^{r,a})\otimes G^{r,a}(x,y) &= \delta(x-y) && (8)\\
(G_0^{-1}-\Sigma^{r})\otimes G^{<,>}(x,y) &= \Sigma^{<,>}\otimes G^{a}(x,y), && (9)
\end{aligned}
$$

where $G_0^{-1} = \partial^2 - m^2$ is the inverse of the free propagator and $\otimes$ represents the convolution integral: $A\otimes B(x,y) \equiv \int d^4z A(x,z)B(z,y)$. Functions (4-7) are not all independent. In fact they can be reduced to just two independent real functions by using (i) the hermiticity constraint $G^r(x,y)^* = G^a(y,x)$ and $G^<(x,y)^* = -G^<(y,x)$, (ii) the relation $G^r - G^a = G^> - G^<$ and (iii) the spectral representation which relates $\mathrm{Im}\, G^{r,a}$ to $\mathrm{Re}\, G^{r,a}$.

Transforming to the Wigner representation, $G(k;X) = \int d^4r e^{ik\cdot r}G(X+r/2, X-r/2)$, and making use of the relation $(\Sigma\otimes G)(k;X) = \exp -i\diamond\{\Sigma(k;X)\}\{G(k;X)\}$, the real part

of (8) becomes an equation for the propagator

$$\cos\diamond\{\Omega^2 \pm i\omega\Gamma\}\{G^{r,a}\} = 1 \tag{10}$$

and the real part of the equation (9) yields the quantum Boltzmann equation for the dynamical variable $G^<$:

$$-\sin\diamond\{\Omega^2\}\{iG^<\}=\frac{1}{2}\cos\diamond(\{\Sigma^>\}\{G^<\} - \{\Sigma^<\}\{G^>\}) - \sin\diamond\{i\Sigma^<\}\{G_R\}, \tag{11}$$

where $\diamond$ is the Poisson bracket operator

$$\diamond\{f\}\{g\} = \frac{1}{2}\left[\partial_X f\cdot\partial_k g - \partial_k f\cdot\partial_X g\right] \tag{12}$$

and we defined the shorthand notation

$$\Omega^2 = k^2 - m^2 - \Sigma_R. \tag{13}$$

The retarded and advanced operators were decomposed as

$$G^{r,a} = G_R \mp i\mathcal{A}\,, \qquad \Sigma^{r,a} = \Sigma_R \mp i\omega\Gamma, \tag{14}$$

where $\mathcal{A}$ is the spectral function. Finally, assuming *spectral decomposition* of the Wigner functions and making use of $\mathcal{A} = i(G^> - G^<)/2$, we can write

$$iG^< = 2\mathcal{A}n \qquad iG^> = 2\mathcal{A}(n+1). \tag{15}$$

Eqs. (10–11) are the full dynamical equations in the Wigner representation. These are suitable for the description of systems in slowly varying backgrounds, with truncation at a given order in gradients leading typically to a more accurate modeling of the dynamics.

2.1 Propagator equation

To the lowest order in gradients Eq. (10) defines the familiar propagators in a spatially constant background. At this order the spectral function $\mathcal{A}$ is singular in the limit $\Gamma \to 0$, and defines the well known projection to the local quasiparticle on-shell. Given the decomposition (15), the QBE (11) also becomes singular, allowing a reduction of the QBE by integration

over momentum (or frequency) to the well known semiclassical Boltzmann equation involving on-shell excitations only [4]. When higher order gradient corrections are included, the on-shell projection becomes more involved because of the coherent quantum effects described by the gradient terms [2]. Before performing the on-shell projection for the QBE, we shall here illustrate the technique using a simple test function.

Solving Eq. (10) iteratively to the lowest nontrivial order in gradients around the lowest order pole gives

$$G^{r,a} \to G_2 = \frac{1}{z} + \frac{1}{2}\frac{m^{2\prime\prime}}{z^3} - \frac{1}{2}\frac{2k_0^2 m^{2\prime\prime} + (m^{2\prime})^2}{z^4}, \tag{16}$$

where $z = k_0^2 - k_x^2$ with $k_0^2 = \omega^2 - \vec{k}_\parallel^{\,2} - m^2(x)$ and we have assumed $\Sigma_R = 0$ for simplicity. The spectral integral over the momentum variable k of some function $\mathcal{T}$ can be converted to a contour integral encircling the multiple pole of G at $z = 0$ [2]:

$$\begin{aligned} \mathcal{I}_p[\mathcal{T}] \;&\equiv\; \frac{2}{\pi}\int_0^\infty dk_x \mathcal{A}_p \mathcal{T} \\ &\to\; \mathrm{Res}\left[G_p \mathcal{T}/\sqrt{k_0^2 - z}\right]_{z=0} = \sum_{i=0}^{p+1} c_i \mathcal{T}^{(i)}(k_0), \end{aligned} \tag{17}$$

where $\mathcal{T}^{(i)}(k_0) = (\partial_{k_x}^i \mathcal{T})(k_0)$, and the coefficients c_i may contain gradients up to pth order with respect to x. To the leading order in gradients ($p = 0$) the spectral function indeed becomes singular: $\mathcal{A} \to (\pi/2k_0)[\delta(k_x - k_0) + \delta(k_x + k_0)]$, projecting sharply on-shell $k_x = \pm k_0$.

It should be noted that the singularity of the propagator (16) remains at the quasiparticle shell $k_x^2 = k_0^2$. The effect of gradient corrections in G_p is to project out derivatives of $\mathcal{T}$ *w.r.t.* k_x up to order $p + 1$. That this produces an effective shift of the pole emerges when the contribution from the first order derivative is included in the definition of the projected function. To second order in gradients

$$\mathcal{I}_2[\mathcal{T}] = \frac{\mathcal{T}(k_{\rm sc})}{k_{\rm sc}} + c_2 \mathcal{T}^{(2)}(k_0) + c_3 \mathcal{T}^{(3)}(k_0), \tag{18}$$

where

$$k_{\rm sc} = k_0 + \frac{1}{8}\frac{m^{2\prime\prime}}{k_0^3} + \frac{5}{32}\frac{(m^{2\prime})^2}{k_0^5} \tag{19}$$

is the space dependent semiclassical momentum which coincides with the standard WKB dispersion relation. The higher order derivative corrections in Eq. (18), however, are of the same order in ∂_x as the shift. As we will now see in detail this leads to the breakdown of the

quasi-particle approximation when the projection is performed on the QBE. It is then not sufficient to describe the system with a single distribution function obtained by projecting on-shell, but some off-shell information, represented by a finite number of derivatives of n, describing coherent quantum effects, is necessary.

3 Quantum Boltzmann Equation

We will now consider the QBE (11) to the lowest nontrivial order in gradients. We truncate the collision term at leading order in gradients and focus on the structure of the flow term in the presence of a varying background. This approximation amounts to neglecting the derivatives of the self-energies, while retaining those of the background. (A complete consideration that includes all second order terms in the collision term will be given elsewhere [6]). We then have

$$-\left(\diamond - \frac{1}{6}\diamond^3\right)\{\Omega^2\}\{iG^<\} = \frac{1}{2}[\Sigma^> G^< - \Sigma^< G^>]. \tag{20}$$

The on-shell projection of the quantum Boltzmann equation is usually done by integrating over frequencies, resulting in an equation on the phase space $\{\vec{k}; t, \vec{x}\}$ [5]. An alternative yet equivalent approach of integrating over momenta is more convenient here, because in the present spatially varying problem the density of states is most conveniently labeled by the conserved energy.

Let us continue with the special case $\Sigma_R = 0$. Inserting the decomposition (15) into (20) and writing the $\diamond$-terms explicitly gives

$$\left(\omega\partial_t + \vec{k}\cdot\partial_{\vec{x}} + \frac{1}{2}(\partial_x k_0^2)\partial_{k_x} - \frac{1}{48}(\partial_x^3 k_0^2)\partial_{k_x}^3\right)\mathcal{A}n = -\frac{1}{2}\mathcal{A}\left(i\Sigma^> n - i\Sigma^<(n+1)\right). \tag{21}$$

Integrating (21) over k_x gives a dynamical equation coupling $n(k_0)$ and its first three derivatives. The information contained in this *zeroth moment* clearly does not suffice to define a closed solution to the problem. To provide closure we can perform integrals of Eq. (21) weighted by some higher powers of k_x, in a manner analogous to the standard derivation of fluid equations by taking moments of the classical Boltzmann equation. There is however one crucial difference. While for the latter case there is no control parameter and hence no natural closure exists, in the former case the equations close at a finite number of independent moments. This is the case because, as shown in equation (17) above, in an integral over

any smooth test function weighted by G_p only the first $p+2$ terms are nonzero. In particular at second order in gradient expansion the closure is obtained by the first four moments.

The QBE becomes very complicated when written in terms of $n^{(i)}(k_0)$. It is more convenient to express the equations in terms of the weighted projections of the generalized distribution function n

$$f_l = \theta(k_x) f_l^+ + \theta(-k_x) f_l^- , \tag{22}$$

where

$$f_l^+ \equiv \frac{2}{\pi} \int_0^\infty dk_x k_x^l \mathcal{A} n \,, \qquad f_l^- \equiv \frac{2}{\pi} \int_{-\infty}^0 dk_x (-k_x)^l \mathcal{A} n. \tag{23}$$

These functions are a straightforward generalization of the distribution function for the spatially constant case. In the absence of gradient corrections $f_l^\pm \to k_0^{l-1} n(\pm k_0)$, so that the standard distribution function is then simply $\theta(k_0)n(k_0) + \theta(-k_0)n(-k_0)$. Now using moments (23) it is particularly easy to prove the closure. Indeed, at second order in gradients $\mathcal{A} \to \mathcal{A}_2$, and we have the identity

$$\int_0^\infty dk_x (k_x - k_0)^4 k_x^l \mathcal{A}_2 n = 0, \tag{24}$$

and the analogous identity holds for $k_x < 0$, so that one immediately obtains

$$f_{l+4} - 4k_0 f_{l+3} + 6k_0^2 f_{l+2} - 4k_0^3 f_{l+1} + k_0^4 f_l = 0. \tag{25}$$

The constraint (25) allows any moment f_l to be written in terms of the four lowest ones f_0, f_1, f_2, f_3. A similar binomial constraint holds at p-th order in gradients, providing closure with $p+2$ moments.

It is curious to observe that the functions f_l can be interpreted as projections onto different momentum hypersurfaces $\tilde{k}_l$ of n. Performing the k-integral in Eq. (23) implies that

$$f_l^\pm = k_0^{l-2} \, \kappa_l \, n(\pm \tilde{k}_l) + \mathcal{O}(\partial_{k_x}^2 n)_{k_x = \pm k_0} , \tag{26}$$

where the lth shell momentum $\tilde{k}_l$ is given by

$$\tilde{k}_l = k_0 + \frac{(l-1)(l-2)}{16} \frac{m^{2\prime\prime}}{k_0^3} + \frac{l^2 - 5l + 5}{32} \frac{(m^{2\prime})^2}{k_0^5} \tag{27}$$

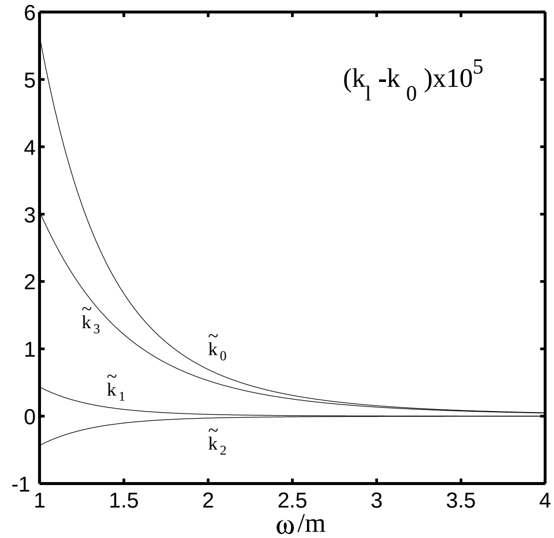

Figure 2: The hypersurfaces $k_x = \tilde{k}_l$ on which the moments f_l flow according to Eq. (29).

and

$$\kappa_l = k_0 + \frac{(l-1)(l-2)(l-3)}{48}\,\frac{m^{2\prime\prime}}{k_0^3} + \frac{(l-1)(l-3)(l-5)}{96}\,\frac{(m^{2\prime})^2}{k_0^5}. \tag{28}$$

In particular $\tilde{k}_0 = k_0^2/\kappa_0 = k_{\rm sc}$ is the semiclassical shell represented by the dispersion relation (19). One thus arrives at the following intuitive picture, illustrated in Fig. 2: taking moments of the QBE (11) corresponds to discretizing momenta with a finite set of hypersurfaces as given by Eq. (26), the flow along which is described by the associated distribution function f_l.

After these considerations it is easy to show that the lth moment of QBE (11) becomes

$$\omega\partial_t f_l + \partial_x f_{l+1} - \frac{l}{2}(\partial_x k_0^2) f_{l-1} + \frac{l(l-1)(l-2)}{48}(\partial_x^3 k_0^2) f_{l-3} = Coll_l\,, \tag{29}$$

where we dropped the term $\vec{k}_\| \cdot \partial_{\vec{x}_\|} f_l$ for simplicity (it can be always reinserted by the replacement $\omega\partial_t \to \omega\partial_t + \vec{k}_\| \cdot \partial_{\vec{x}_\|}$). The first four equations (29), with $l = 0, 1, 2, 3$, together with the closure condition (25), form a closed set for functions f_0, f_1, f_2 and f_3. These equations are our main result. They can be used as a starting point for studying plasma dynamics in out-of-equilibrium situations when gradient approximation applies.

Given the simple 4-point interaction term depicted in Fig. 1, the lth moment of the

collision term appearing in (29) becomes

$$Coll_l = -\Gamma_l^{>} f_l + \Gamma_l^{<}\left(f_l + k_0^{l-2}\kappa_l\right) \tag{30}$$

where $\Gamma_l^{>}$and $\Gamma_l^{<}$ are the spectral projections of the self-energies $i\Sigma^{>}$ and $i\Sigma^{<}$ defined by

$$\begin{aligned}
\Gamma_l^{>} &= \frac{\lambda^2}{2}\int_{p,k',p'}(2\pi)^4\delta^4(k+p-k'-p')f_p(f_{p'}+1)(f_{k'}+1)\\
\Gamma_l^{<} &= \frac{\lambda^2}{2}\int_{p,k',p'}(2\pi)^4\delta^4(k+p-k'-p')(f_p+1)f_{p'}f_{k'},
\end{aligned} \tag{31}$$

where $\int_p \equiv \int d\omega d^2p_{\|}/[(2\pi)^3 2p_{\rm sc}]$ is the semiclassical equivalent of the Lorentz invariant three dimensional measure, and the x-component of the δ-function should be taken as $\delta(\theta(k_x)\tilde{k}_l + \theta(p_x)p_{\rm sc} - \theta(k'_x)k'_{\rm sc} - \theta(p'_x)p'_{\rm sc})$. However, by the same approximation we made to arrive to Eqn. (20), we can set $\tilde{k}_l \to k_{sc}$ in the delta function, which immediately leads to the identification $\Gamma_l^{<,>} = \Gamma_0^{<,>} \equiv \Gamma^{<,>}$. We will see below (*cf.* Eq. (41)) how, after a particular change of variables, $\Gamma^{<,>}$ reduces to a more canonical form. In the more general case, when second order gradients are included, $Coll_l$ depends on moments f_l, f_{l-1}, f_{l-2} and is a functional of f_0, but not of higher order moments, so that including the gradient corrections to $Coll_l$ would not spoil the closure property.

4 Quantum coherence and semiclassical BE

In order to separate the effects of quantum coherence in the dynamical equations (29), it is convenient to define the following linear combinations of f_l:

$$\begin{aligned}
f &= k_{\rm sc}f_0\\
f_{\rm qc1} &= f_1 - f\\
f_{\rm qc2} &= f_2/\kappa_2 - f_{\rm qc1} - f\\
f_{\rm qc3} &= f_3/k_0^2 - f_{\rm qc2} - f_{\rm qc1} - f.
\end{aligned} \tag{32}$$

The distribution function f measures by definition population density of particles on the hypersurface $k_x = \tilde{k}_0 \equiv k_{\rm sc}$, while the densities $f_{\rm qci}$ measure the correlations (quantum coherence) between neighboring hypersurfaces $k_x = \tilde{k}_{i-1}$ and $k_x = \tilde{k}_i$, where $\tilde{k}_i$ are defined in Eq. (27). Note that in equilibrium in a uniform background all $f_{\rm qci}$ vanish.

Multiplying Eq. (29) for f_0 by $k_{\rm sc}/\omega$ one obtains

$$\partial_t f + \frac{k_{\rm sc}}{\omega}\partial_x(f + f_{\rm qc1}) = -\frac{\Gamma^>}{\omega} f + \frac{\Gamma^<}{\omega}(f+1). \tag{33}$$

This equation already resembles the standard Boltzmann equation, the main difference being coupling to an unknown function $f_{\rm qc1}$. Dividing the f_1-equation by ω and subtracting Eq. (33) one then obtains

$$(\partial_t + \Gamma_{\rm qc})f_{\rm qc1} + \frac{\kappa_2 - k_{\rm sc}}{\omega}\partial_x f_{\rm qc1} + \frac{\partial_x\kappa_2}{\omega}(f_{\rm qc1} + f_{\rm qc2}) + \frac{\kappa_2}{\omega}\partial_x f_{\rm qc2} = s_1, \tag{34}$$

where the source

$$s_1 = \frac{k_{\rm sc} - \kappa_2}{\omega}\partial_x f + \frac{(\partial_x k_0^2/2k_{\rm sc}) - \partial_x\kappa_2}{\omega} f \tag{35}$$

represents coherent mixing of f and $f_{\rm qc1}$. One can obtain similar equations for the coherent quantum densities $f_{\rm qc2}$ and $f_{\rm qc3}$, but we will not present them explicitly here. The coherent density $f_{\rm qci}$ is damped at the rate $\Gamma_{\rm qc}$ which reads

$$\Gamma_{\rm qc} = \frac{\Gamma^> - \Gamma^<}{\omega}. \tag{36}$$

This is the out-of equilibrium generalization of the on-shell damping rate (*cf.* Eq. (31) and [7]). The coherence equations for $f_{\rm qci}$ are linear and hence can be quite easily solved, and the solution for $f_{\rm qc1}$ inserted into Eq. (33). Requiring that none of $f_{\rm qci}$ be sourced by the self-energy $\Gamma^<$ defines $f_{\rm qci}$ uniquely.

We now pause to discuss the validity of the gradient approximation. It is not hard to see that in the model under study the validity criteria reduce to the following conditions

$$\partial_t f \ll \omega f, \qquad \partial_x f \ll k_0 f, \tag{37}$$

which are the particular realization of $||\Diamond|| \ll 1$ for the on-shell Boltzmann equation.

In order to study how quantum coherence influences Eq. (33), we now make a simple estimate of $f_{\rm qci}$. Not far from equilibrium f can be approximated by its equilibrium form so that $\partial_x f \sim m^2/\omega L T$, where L represents the scale on which m^2 varies. To leading order in m^2 the source s_1 in Eq. (35) can be estimated as

$$s_1 \sim \frac{1}{(Lk_0)^3}\frac{m^2}{\omega}. \tag{38}$$

Similar estimate holds for s_2 and s_3 in the equations for $f_{\rm qc2}$ and $f_{\rm qc3}$. If we want to include the higher order derivatives of $k_{\rm sc}$ in Eq. (33), this estimate implies that one cannot in general neglect $f_{\rm qc1}$ in Eq. (33).

There is however a limit in which it is legitimate to neglect the part of $f_{\rm qc1}$ in Eq. (33) sourced by s_1 and still maintain higher order derivatives in $k_{\rm sc}$. Indeed, assuming *efficient scattering*, so that coherent quantum densities are strongly damped and the derivative terms in Eq. (34) can be neglected, we arrive at the following estimate of the coherent quantum densities sourced by s_1:

$$f_{\rm qci} \sim \frac{1}{\Gamma_{qc}L}\frac{1}{(Lk_0)^2}\frac{m^2}{\omega k_0} \qquad (i = 1, 2, 3). \tag{39}$$

To check consistency of this estimate, note that in the efficient scattering limit one expects $\partial_x f_{\rm qci} \sim 1/L$, so that the spatial derivative term is suppressed by $1/(\Gamma_{\rm qc}L)$ in comparison to the $\Gamma_{\rm qc} f_{\rm qc1}$ term. In the stationary frame the time derivative term is suppressed by an additional plasma velocity, which is often much smaller than unity. For example, the phase boundary speed at the electroweak phase transition is smaller than about 0.3 of the speed of light [8]. Upon inserting Eq. (39) into Eq. (33), we see that the coherent quantum density $f_{\rm qc1}$ is suppressed by $1/(\Gamma_{\rm qc}L)$ in comparison to the other third order gradient terms in Eq. (33). We have thus shown that in the limit $\Gamma_{\rm qc}L \gg 1$ the coherent quantum density $f_{\rm qc1}$ in Eq. (33) can be neglected so that we arrive at the semiclassical Boltzmann equation

$$\partial_t f + \frac{k_{\rm sc}}{\omega}\partial_x f = Coll[f], \tag{40}$$

where, at the leading order in gradients, the collision term is given in Eq. (33). Note that this equation still does not have the standard form since we have integrated over momentum k_x and thus obtained a distribution function $f = f(\omega, \vec{k}_\parallel; t, x) = k_{\rm sc} f_0$ which is a function of energy ω. To recover the semiclassical Boltzmann equation on the usual phase space we make the change of variables defined as follows

$$\frac{d\omega}{k_{\rm sc}(\omega; x)} \to \frac{dk_x}{\tilde{\omega}(k_x; x)}. \tag{41}$$

This choice is natural because it gives the local Lorentz covariant measure in the collision integral (31). This change of variables in fact does not leave $\delta^4(k + p - k' - p')$ invariant.

The correction induced is, however, second order in derivatives of the distribution function moments in the collision term, and hence it is beyond the approximation considered in this letter. Making use of $d\omega \to d\tilde{\omega}(k_x;x) = (\partial_{k_x}\tilde{\omega})dk_x$, we then immediately obtain

$$\partial_{k_x}\tilde{\omega}(k_x;x) = \frac{k_{\rm sc}(\tilde{\omega}(k_x;x);x)}{\tilde{\omega}(k_x;x)}, \tag{42}$$

which specifies $\tilde{\omega} = \tilde{\omega}(k_x;x)$. Consider now how the flow term transforms under this change of variables. We first have $f(\omega;x,t) \to f(k_x;x,t)$, so that

$$\begin{aligned} \frac{k_{\rm sc}}{\omega} &\to v_g \equiv \partial_{k_x}\tilde{\omega} \\ \partial_x &\to \partial_x - (\partial_{k_x}\tilde{\omega})^{-1}(\partial_x\tilde{\omega})\partial_{k_x}, \end{aligned} \tag{43}$$

and hence Eq. (40) becomes

$$\partial_t f + v_g \partial_x f + F_x \partial_{k_x} f = Coll[f], \tag{44}$$

with the following canonical velocity and force

$$\begin{aligned} v_g &= \partial_{k_x}\tilde{\omega} \\ F_x &= -\partial_x\tilde{\omega}. \end{aligned} \tag{45}$$

This semiclassical Boltzmann equation is one of our main results. Note that "inversion" of the semiclassical dispersion relation $k_{\rm sc} = k_{\rm sc}(\omega;x)$ as given in Eq. (42) is defined in a rather non-trivial way by invoking a natural transformation of the collision integral measure, resulting in a different form of the semiclassical energy hypersurface $\tilde{\omega} = \tilde{\omega}(k_x;x)$ from what one might naively guess. This is the unique choice which, within the present approximation, renders the canonical form for the semiclassical velocity and force (45). Recall again however, that in the present work the $\diamond^2$-term operating on the collision term and other analogous contributions were neglected. In a completely consistent computation these terms must be retained, and we are currently pursuing this computation in [6]. We finally emphasize that in case when the frequent scattering limit does not apply, one has to use more general equations (29) with the closure relation (25).

5 Discussion

We have generalized the semiclassical Boltzmann equation to include the effects of a slowly varying background to nontrivial order in gradients for a complex scalar field. Our motivation is baryon production at a first order electroweak phase transition, where this generalization is necessary to model CP violating effects which first appear in transport equations beyond leading order in gradient expansion. We found that consistent treatment of the system to nontrivial order in gradients requires the introduction of coherent quantum densities in addition to the usual distribution function, and derived the corresponding dynamical equations. We also showed that in the limit of frequent scatterings coherent quantum effects are suppressed, and the problem reduces to a single Boltzmann equation for the distribution function, containing a classical force which includes gradient corrections to nontrivial order. This is a particularly nice realization of the quantum-to-classical transition which occurs by the dissipative dynamics within the system itself. The dynamics is rendered dissipative by the weak coupling truncation of the Schwinger-Dyson equation at the order λ^2, and by the truncation of the gradient expansion at third order in gradients, resulting in localized dynamics in the Wigner representation and irreversibility. The rate at which the coherent quantum effects are dissipated is simply the out-of-equilibrium damping rate (36).

We now comment on how to relate our results to electroweak baryogenesis. The relevant CP violating effects emerge generally only at non-trivial order in gradients of the background, which is analogous to the case considered here. To make a connection with the computation of baryogenesis sources, it is instructive to integrate equation (33) over ω and $\vec{p}_\parallel$ to obtain a continuity equation for the density of particles which contains clearly separated semiclassical and quantum mechanical sources of the form

$$j_{\rm sc} = \int \frac{d\omega d^2 p_\parallel}{(2\pi)^3} \frac{k_{\rm sc}}{\omega} f, \qquad j_{\rm qc} = \int \frac{d\omega d^2 p_\parallel}{(2\pi)^3} \frac{k_{\rm sc}}{\omega} f_{\rm qc1}. \tag{46}$$

This should be contrasted with the situation in literature [9], [10], [11], where there is no agreement on whether the relevant CP violating sources come from classical or quantum currents. Moreover, in the frequent scattering limit, $\Gamma_{qc} L >> 1$, we find that the coherent quantum current $j_{\rm qc}$ is suppressed, and the (dominant) semiclassical current then should be calculated making use of the semiclassical transport equation (44). This contains a classical

force analogous to the one first introduced as a source for baryogenesis in studies of two Higgs doublet models in Ref. [9], and then subsequently applied to the MSSM in Ref. [12].

Here we studied for simplicity a complex scalar field, whereas the physically relevant cases involve mixing scalar or fermion fields. However we believe that the novel features we found, being essentially a consequence of localization in space, are not specific to the complex scalar theory, but generic to all models. How this is realized in detail in other theories is under investigation [5, 6].

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